

# Nuclear spin dynamics and Zeno effect in quantum dots and defect centers

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We analyze nuclear spin dynamics in quantum dots and defect centers with a bound electron under electron-mediated coupling between nuclear spins due to the hyperfine interaction (“J-coupling” in NMR). Our analysis shows that the Overhauser field generated by the nuclei at the position of the electron has short-time dynamics quadratic in time for an initial nuclear spin state without transverse coherence. The quadratic short-time behavior allows for an extension of the Overhauser field lifetime through a sequence of projective measurements (quantum Zeno effect). We analyze the requirements on the repetition rate of measurements and the measurement accuracy to achieve such an effect. Further, we calculate the long-time behavior of the Overhauser field for effective electron Zeeman splittings larger than the hyperfine coupling strength and find, both in a Dyson series expansion and a generalized master equation approach, that for a nuclear spin system with a sufficiently smooth polarization the electron-mediated interaction alone leads only to a partial decay of the Overhauser field by an amount on the order of the inverse number of nuclear spins interacting with the electron.

## I. INTRODUCTION

Technological advancements have made it possible to confine very few electrons in a variety of nanostructures such as nanowires, quantum dots, donor impurities, or defect centers.<sup>1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22</sup>

One driving force behind these achievements is a series of proposals for using the spin of an electron as a qubit for quantum computing.<sup>23,24,25</sup> This spin interacts with the nuclear spins in the host material via the hyperfine interaction. While this interaction leads to decoherence of the electron spin state on one hand, it also provides the opportunity to create a local effective magnetic field (Overhauser field) for the electron by inducing polarization in the nuclear spin system, which could be used, e.g., for rapid single-spin rotations.<sup>26</sup> Polarizing the nuclear spin system is also one possible way to suppress hyperfine-induced decoherence<sup>27</sup> or it can be used as a source of spin polarization to generate a spin-polarized current. In any case, controlling the dynamics of the Overhauser field and, in particular, to prevent its decay, is thus of vital importance in the context of spintronics and quantum computation.<sup>28</sup>

In GaAs quantum dots the Overhauser field can become as large as 5T. The build-up, decay, and correlation time of the Overhauser field have been studied in a number of systems,<sup>29,30,31,32,33,34,35,36,37,38,39,40,41,42</sup> suggesting timescales for the decay on the order of seconds, minutes, or in one case, even hours.<sup>43</sup>

The dynamics of the Overhauser field are governed by the mutual interaction between the nuclear spins. There is on one hand the direct dipolar coupling between the nuclear spins. On the other hand, due to the presence of a confined electron, there is also an indirect interaction: The coupling of the nuclear spins to the electron via the hyperfine interaction leads to an effective interaction

between the nuclear spins that is known as the electron-mediated interaction. While the effect of this electron-mediated interaction on the decoherence of the *electron* has been studied previously,<sup>44,45,46</sup> theoretical studies of the decay of the Overhauser field have so far studied direct dipole-dipole interaction and the effect of the hyperfine interaction was taken into account through the Knight shift that the electron induces via the hyperfine interaction.<sup>47</sup> In this article we investigate the effect of the electron-mediated interaction between nuclear spins on the dynamics of the Overhauser field. While the direct dipolar coupling is always present, it can be weaker than the electron-mediated interaction for magnetic fields that are not too large and may be further reduced via NMR pulse sequences or by diluting the concentration of nuclear spins.<sup>48</sup> We find in our calculation that, for effective electron Zeeman splittings  $\omega$  (sum of Zeeman splittings due to the external magnetic field and Overhauser field) larger than the hyperfine coupling strength  $A$ , the decay of the Overhauser field due to the electron-mediated interaction is incomplete, i.e., that only a small fraction of the Overhauser field decays. In a short-time expansion that is valid for  $\omega$  larger than  $A/\sqrt{N}$ , where  $N$  is the number of nuclear spins with which the electron interacts, we find a quadratic initial decay on a timescale  $\tau_e = N^{3/2}\omega/A^2$ . We show that, by performing repeated projective measurements on the Overhauser field, a quantum Zeno effect occurs, which allows one to preserve the Overhauser field even for relatively small effective electron Zeeman splittings larger than  $A/\sqrt{N}$ .

In Sec. II we briefly review the quantum Zeno effect and give the corresponding main results for the case of the Overhauser field. We start our detailed discussion in Sec. III by writing down the Hamiltonian for the hyperfine interaction and by deriving an effective Hamiltonian for the electron-mediated interaction. In Sec. IV we derive an expression for the short-time behavior of the

Overhauser field mean value. In Secs. V and VI we address the long-time decay of the Overhauser field due to the electron-mediated interaction. Some technical details are deferred to Appendices A and B.

## II. ZENO EFFECT

The suppression of the decay of a quantum state due to frequently repeated measurements is known as the quantum Zeno effect. The concept of the quantum Zeno effect<sup>49</sup> is almost as old as quantum mechanics<sup>50,51</sup> and it remains one of the most intriguing quantum effects. It has been studied intensively from the theoretical side<sup>52,53,54</sup> and also experimental evidence has been found in recent years.<sup>55</sup>

For a two-level system initialized to the excited state, the survival probability  $P_s$  in the excited state as a function of the elapsed time  $t$  is initially given by  $P_s(t) = 1 - c_s t^2 / \tau_s^2$ , with the constant  $c_s$  and the timescale  $\tau_s$  being system dependent. A projective measurement at time  $\tau_m$  resets the system to the excited state with probability  $P_s(\tau_m)$ . Repeating the measurement  $m$  times at intervals  $\tau_m \ll \tau_s$ , the survival probability is  $P_{s,meas}(m\tau_m) = (1 - c_s \tau_m^2 / \tau_s^2)^m \approx 1 - c_s m \tau_m / (\tau_s^2 / \tau_m)$ , for  $c_s m \tau_m^2 / \tau_s^2 \ll 1$ . The survival probability at time  $t = m\tau_m$  is thus increased due to the frequently repeated measurements: instead of a quadratic decay on a timescale  $\tau_s$  without measurements, we have a linear decay on a timescale  $\tau_s^2 / \tau_m$ .

A more complex observable such as the mean of the Overhauser field  $\langle h_z(t) \rangle = \text{Tr}\{h_z \rho(t)\}$ , may also show a Zeno effect. That  $\langle h_z(t) \rangle$  shows an initial quadratic decay is, however, not obvious and actually depends on the initial state of the nuclear spin system  $\rho_I(0)$ . For the short-time behavior of  $\langle h_z(t) \rangle$  we expand in a Taylor series

$$\langle h_z(t) \rangle = \langle h_z(0) \rangle + t \langle h_z \rangle_1 + \frac{t^2}{2} \langle h_z \rangle_2 + \dots, \quad (1)$$

with  $\langle h_z \rangle_n = d^n \langle h_z(t) \rangle / dt^n|_{t=0}$ . If  $\langle h_z \rangle_1 = 0$ , the  $t$ -linear term vanishes and the initial decay is quadratic in time. In Sec. IV we calculate the initial dynamics of  $\langle h_z(t) \rangle$  and explain the conditions under which  $\langle h_z \rangle_1 = 0$ . We find an initial decay of the form

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - c \frac{t^2}{\tau_e^2}. \quad (2)$$

The timescale  $\tau_e$  and the constant  $c$  are given below in Eq. (17) and Eq. (18) respectively.

Let us now consider a sequence of repeated measurements of the Overhauser field  $h_z(t)$ . In the context of quantum dots, several proposals<sup>56,57,58</sup> to implement such measurements have been put forward. A measurement of  $h_z$  shall be performed after a time  $\tau_m$ . If this measurement is projective, i.e., if it sets all the off-diagonal elements of the density matrix in a basis of  $h_z$ -

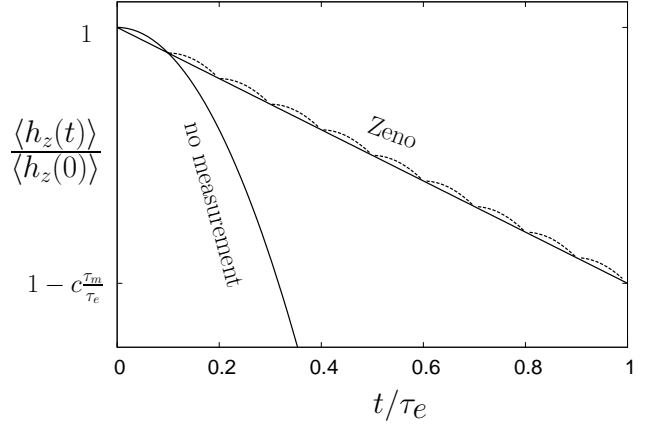


FIG. 1: Effect of projective measurements at time intervals  $\tau_m = \tau_e/10$  on the time evolution of the Overhauser field expectation value  $\langle h_z(t) \rangle$ . Due to the Zeno effect, the decay with measurements is  $1 - ct/\tau_{zeno}$  rather than  $1 - ct^2/\tau_e^2$  without measurements, where  $\tau_{zeno} = \tau_e^2/\tau_m$ . The formula  $1 - ct/\tau_{zeno}$  for the decay with measurement is only strictly valid at times  $t = m\tau_m$  with  $m$  being a positive integer. After the measurement at  $t = m\tau_m$  the decay is again quadratic with time dependence  $\langle h_z(m\tau_m) \rangle / \langle h_z(0) \rangle - c(t - m\tau_m)^2/\tau_e^2$  (broken lines).

eigenstates to zero (we discuss requirements on the accuracy of the measurement in Appendix B), the dynamics after  $\tau_m$  again follow Eq. (2). Repeating the measurement at times  $2\tau_m, 3\tau_m, \dots$ , leads to a change of the decay of the Overhauser field in the same way as we described it for the two-level system above:

$$\frac{\langle h_z(t) \rangle_{zeno}}{\langle h_z(0) \rangle} = 1 - c \frac{t}{\tau_{zeno}}, \quad \tau_{zeno} = \frac{\tau_e^2}{\tau_m}. \quad (3)$$

Instead of a quadratic decay  $\propto t^2/\tau_e^2$  we have a linear decay  $\propto t/\tau_{zeno}$  with  $\tau_{zeno} = \tau_e^2/\tau_m$ . We note that the expression for  $\langle h_z(t) \rangle_{zeno}$  in Eq. (3) is only strictly valid at times  $m\tau_m$  with  $m$  being a positive integer. Between these times  $\langle h_z(t) \rangle$  changes according to Eq.(2). The derivation of Eq. (3) requires  $cm\tau_m^2/\tau_e^2 = ct/\tau_{zeno} \ll 1$ . Fig. 1 shows the Zeno effect, i.e., the difference between  $\langle h_z(t) \rangle / \langle h_z(0) \rangle$  and  $\langle h_z(t) \rangle_{zeno} / \langle h_z(0) \rangle$ .

In addition to requirements on the measurement accuracy (see Appendix B), the results in this section rest on the following separation of timescales:

$$\tau_{pm} \ll \tau_m \ll \tau_e, \tau_x, \quad (4)$$

where  $\tau_{pm}$  is the time required to perform a single measurement and  $\tau_x$  the timescale up to which the short-time expansion for  $\langle h_z(t) \rangle$  is valid. In general,  $\tau_x$  can be shorter than  $\tau_e$ . A specific case (fully polarized nuclear state), where the short-time expansion has only a very limited range of validity, is discussed in Sec. IV A. For the systems studied in experiment, we expect  $\tau_x$  to be

comparable to or longer than  $\tau_e$ , since the experiments performed so far show timescales for the decay of  $\langle h_z(t) \rangle$  on the order of seconds, minutes, or in one case, even hours.<sup>43</sup> We note that it may be a demanding task to perform the fast and precise measurements required to obtain a Zeno effect in the present context. Still, experimental progress in the control of the nuclear field, such as that shown in Ref. 43, suggests that such measurements may be within reach in the near future.

We continue our discussion by deriving the effective Hamiltonian we use both for calculating short-time dynamics and the long-time behavior of  $\langle h_z(t) \rangle$ .

### III. HAMILTONIAN

We aim to describe the dynamics of many nuclear spins surrounding a central confined electron spin in a material with an s-type conduction band (e.g. GaAs, Si, etc.), where the dominant type of hyperfine interaction is the Fermi contact hyperfine interaction. The electron may be confined in many nanostructures such as nanowires, quantum dots or defect centers. Under the assumption that other possible sources of nuclear spin dynamics, such as nuclear quadrupolar coupling, are suppressed,<sup>59</sup> the two strongest interactions between nuclear spins in these nanostructures are the electron-mediated interaction (“J-coupling” in NMR<sup>60,61</sup>) and the direct dipole-dipole interaction. It turns out that, for a large number of nuclei  $N$  and up to magnetic fields of a few Tesla (for GaAs), the contribution of the electron-mediated interaction to the initial decay of the Overhauser field is dominant (see Appendix A). The Hamiltonian contains three parts: The electron and nuclear Zeeman energies and the Fermi contact hyperfine interaction:

$$H = H_e + H_n + H_{en} = \epsilon_z S_z + \eta_z \sum_k I_k^z + \vec{S} \cdot \vec{h}. \quad (5)$$

Here, the operator

$$\vec{h} = \sum_k A_k \vec{I}_k \quad (6)$$

is the Overhauser field. Further,  $\vec{S}$  is the electron spin and  $\vec{I}_k$  the nuclear spin at lattice site  $k$  that couples with strength  $A_k = A\nu_0|\psi(r_k)|^2$  to the electron spin, where  $A = \sum_k A_k$  is the total hyperfine coupling constant,  $\nu_0$  the volume occupied by a single-nucleus unit cell and  $\psi(r_k)$  the electron envelope wave function. We define the number of nuclear spins  $N$  interacting with the electron as the number of nuclear spins within an envelope-function Bohr radius of the confined electron.<sup>27</sup> The Bohr radius  $a_B$  for an isotropic electron envelope is defined through<sup>27</sup>  $\psi(r_k) = \psi(0)e^{-(r_k/a_B)^q/2}$ , where  $q = 1$  gives a hydrogen-like wave function and  $q = 2$  a Gaussian. Finally,  $\epsilon_z$  and  $\eta_z$  are the electron and nuclear Zeeman splittings, respectively (we consider a homonuclear system). We derive an effective Hamiltonian for

the electron-mediated interaction between nuclear spins, which is valid in a sufficiently large magnetic field. Using a standard Schrieffer-Wolff transformation<sup>62</sup>  $H_{\text{eff}} = e^S H e^{-S}$ , in lowest order in  $H_{en}$ , with the transformation matrix  $S = \sum_k A_k ((\epsilon_z + h_z - \eta_z + A_k/2)^{-1} S_+ I_k^- - (\epsilon_z + h_z - \eta_z - A_k/2)^{-1} S_- I_k^+) / 2$ , which eliminates the off-diagonal terms between electron and nuclear spins, we find the effective Hamiltonian  $H_{\text{eff}} \simeq H_0 + V$  (similar to Refs. 44,46,63), where:

$$H_0 = \epsilon_z S_z + \eta_z \sum_k I_k^z + S_z h_z, \quad (7)$$

$$V = \frac{1}{4(\epsilon_z - \eta_z + h_z)} \left( \{h_-, h_+\} S_z + \frac{1}{2} [h_-, h_+] \right). \quad (8)$$

In Eq. (8) we have neglected terms which are suppressed by a factor  $A_k/(\epsilon_z - \eta_z + h_z)$  and the raising and lowering operators are defined as  $S_{\pm} = S_x \pm iS_y$  and similarly for  $h_{\pm}$  and  $I_k^{\pm}$ . The commutator  $[h_-, h_+]$  is defined in the usual way and  $\{h_-, h_+\} = h_- h_+ + h_+ h_-$  is the anti-commutator of  $h_-$  and  $h_+$ . We note that  $H_{\text{eff}}$  neglects the transfer of spin polarization from the electron to the nuclei. The electron transfers an amount of angular momentum to the nuclear system on the order  $(A/\sqrt{N}\omega)^2 \ll 1$  for  $\omega \gg A/\sqrt{N}$ . For  $\omega \sim A$  these contributions are suppressed by a factor of  $O(1/N)$  compared to the decay of  $\langle h_z(t) \rangle$  under  $H_{\text{eff}}$ . For very special initial states, where  $H_{\text{eff}}$  leads to no dynamics, e.g., for uniform polarization, the transfer of spin from the electron to the nuclei is the only source of nontrivial nuclear spin dynamics and therefore should be taken into account. We discuss one such initial state, namely, a fully polarized nuclear system, in Sec. IV A.

In the following we further replace  $h_z$  in the denominator of Eq. (8) by its initial expectation value  $\langle h_z \rangle = \text{Tr}\{h_z \rho(0)\}$  and introduce the effective electron Zeeman splitting

$$\omega = \epsilon_z - \eta_z + \langle h_z \rangle \approx \epsilon_z + \langle h_z \rangle. \quad (9)$$

This replacement assumes that the initial state does not change significantly and is valid up to corrections suppressed by  $\sigma/\omega$ , compared to the dynamics under  $H_{\text{eff}}$ . Here  $\sigma = \sqrt{\langle h_z^2 \rangle - \langle h_z \rangle^2}$  is the initial width of  $h_z$ . For an unpolarized equilibrium (infinite temperature) nuclear spin state we have  $\sigma \propto A/\sqrt{N}$ , limiting the range of validity to  $\omega \gg A/\sqrt{N}$ . Further restricting our treatment to  $I = 1/2$  we may write  $V$  as

$$V \cong \frac{1}{2\omega} \left( S_z \sum_{k \neq l} A_k A_l I_k^+ I_l^- + \frac{1}{2} \sum_k A_k^2 (S_z - I_k^z) \right), \quad (10)$$

where in the sum over  $k$  and  $l$  the terms  $k = l$  are excluded. In the next sections we will discuss the dynamics of the Overhauser field both at short and at long times in the regimes where a perturbative treatment in  $V$  is appropriate.

#### IV. SHORT-TIME EXPANSION

With respect to the Zeno effect as discussed in Sec. II, our main interest lies in the short-time behavior of  $\langle h_z(t) \rangle$  (see Eq. (1)). To calculate  $\langle h_z \rangle_1$  and  $\langle h_z \rangle_2$ , we expand

$$\langle h_z(t) \rangle = \text{Tr}\{h_z \exp(-iHt)\rho(0) \exp(iHt)\} \quad (11)$$

at short times. The first term  $\langle h_z(0) \rangle = \text{Tr}\{h_z \rho(0)\}$  gives the expectation value at time zero, while the  $t$ -linear term is proportional to  $\langle h_z \rangle_1 = -i \text{Tr}\{h_z [H, \rho(0)]\}$ . Using the cyclicity of the trace we find that  $\text{Tr}\{h_z [H, \rho(0)]\} = \text{Tr}\{[\rho(0), h_z] H\}$ . Writing  $\rho(0) = \rho_e(0) \otimes \rho_I(0)$  we have, for an initial nuclear spin state  $\rho_I(0)$  without transverse coherence,  $[\rho_I(0), h_z] = 0$  and thus the  $t$ -linear term vanishes.

To determine the frequency of projective measurements required to induce a Zeno effect, we are interested in  $\langle h_z \rangle_2 = -\text{Tr}\{h_z [H, [H, \rho(0)]]\}$ . We calculate  $\langle h_z \rangle_2$  below using the effective Hamiltonian  $H_{\text{eff}}$  as derived in Sec. III. The range of validity is limited by higher-order terms in the effective Hamiltonian which are proportional to  $(h_+ h_-)^n / \omega^{(n+1)}$ ,  $n = 2, 4, \dots$ . These higher-order terms give corrections to  $\langle h_z \rangle_2$  which are suppressed by a factor  $(A/\sqrt{N}\omega)^n$ . Thus the results for  $\langle h_z(t) \rangle$  up to  $O(t^2)$  given below are valid in the regime  $\omega \gg A/\sqrt{N}$ . Using that  $[h_z, H_0] = 0$ , we may simplify  $\langle h_z \rangle_2$  considerably and we find for an arbitrary electron spin state:

$$\langle h_z \rangle_2 = -\frac{1}{8\omega^2} \text{Tr}_I\{h_z [\rho_I(0), h_+ h_-] h_+ h_-\}. \quad (12)$$

To further simplify, we assume a product initial state of the form

$$\rho(0) = \rho_e(0) \otimes \rho_I(0) = \rho_e(0) \otimes_k \rho_{I_k}, \quad (13)$$

$$\rho_{I_k} = 1/2 + f_k I_k^z; \quad f_k \equiv f_k(0) = 2\langle I_k^z(0) \rangle. \quad (14)$$

For simplicity we restrict our treatment to  $I = 1/2$  and thus  $f_k \in [-1, 1]$ . With this we find

$$\langle h_z \rangle_2 = -\frac{1}{4\omega^2} \sum_{kl} f_k A_k^2 A_l^2 \text{Tr}_I\{h_z \bigotimes_{j \neq k, l} \left(\frac{1}{2} + f_j I_j^z\right) (I_k^z - I_l^z)\}. \quad (15)$$

Evaluating the commutators and the trace, we find for the decay of the Overhauser field mean value  $\langle h_z(t) \rangle$ , up to corrections of  $O(t^4)$ ,

$$\langle h_z(t) \rangle = \langle h_z(0) \rangle - \frac{t^2}{(8\omega)^2} \sum_{kl} A_k^2 A_l^2 (A_k - A_l) (f_k - f_l). \quad (16)$$

We note that both for uniform coupling constants  $A_k = A/N$  and for uniform polarization  $f_k = p, \forall k$ , the  $t^2$ -term vanishes. This is, in fact, what one would expect, since  $H_{\text{eff}}$  only leads to a redistribution of polarization and both for uniform polarization and uniform coupling constants, such a redistribution does not affect  $h_z$ . Rewriting

the sum in Eq. (16) we obtain (again up to corrections of  $O(t^4)$ )

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - c \frac{t^2}{\tau_e^2}, \quad \tau_e = \frac{N^{3/2} \omega}{A^2}, \quad (17)$$

with the numerical factor  $c$  only depending on the distribution of coupling constants through  $\alpha_k = N A_k / A$  and the initial polarization distribution  $f_k$  through

$$c = \frac{1}{32Nc_0} \sum_{kl} \alpha_k^2 \alpha_l^2 (\alpha_k - \alpha_l) (f_k - f_l), \quad (18)$$

where  $c_0 = \sum_k f_k \alpha_k$ . We note that, up to the factor  $c$  (see Fig. 2), the timescale  $\tau_e$  agrees with a previous rough estimate<sup>57</sup> for the timescale of nuclear-spin dynamics under the electron-mediated nuclear spin interaction. In Table I we give  $\tau_e$  for a variety of values of the number of nuclear spins  $N$  and of  $\omega = \epsilon_z - \eta_z + \langle h_z \rangle$ .

		$\tau_e$ at	$\tau_e$ at	$\tau_e$ at	$\tau_e$ at	$\tau_e$ at
$N$	$A/g\mu_B\sqrt{N}$	$\omega = A/\sqrt{N}$	100mT	1T	2mT	5T
$10^3$	49mT	3ns	6ns	60ns	119ns	297ns
$10^4$	16mT	29ns	188ns	2 $\mu$ s	4 $\mu$ s	9 $\mu$ s
$10^5$	49mT	292ns	6 $\mu$ s	60 $\mu$ s	119 $\mu$ s	297 $\mu$ s
$10^6$	1.6mT	3 $\mu$ s	188 $\mu$ s	2ms	4ms	9ms

TABLE I: This Table gives explicit values for the timescale  $\tau_e$  of the  $t^2$  term in the short-time expansion of  $\langle h_z(t) \rangle$ . We give  $\tau_e$  for various values of the number of nuclear spins  $N$  and of  $\omega = \epsilon_z - \eta_z + \langle h_z \rangle$ . When  $\omega = A/\sqrt{N}$  we are at the lower boundary of  $\omega$ -values for which the result for  $\tau_e$  is valid. The parameters used are relevant for a lateral GaAs quantum dot:  $A = 90\mu\text{eV}$ ,  $g = -0.4$ .

The coupling constants  $A_k$  have a different dependence on  $k$ , depending on the dimension  $d$  and the exponent  $q$  in the electron envelope wave function through<sup>27</sup>  $A_k = A_0 e^{-(k/N)^{q/d}}$ . For a donor impurity with a hydrogen-like exponential wave function we have  $d = 3, q = 1, d/q = 3$ , whereas for a 2-dimensional quantum dot with a Gaussian envelope function we have  $d = 2, q = 2, d/q = 1$ . In Fig. 2 we show the constant  $c$  for the case  $d/q = 1$  and a particular choice of the polarization distribution. We give the dependence on  $d/q$  in the inset of Fig. 2. While  $c$  is independent of  $N$  for  $N \gtrsim 100$ , it changes considerably depending on the initial nuclear spin state, which is parameterized by the  $f_k$ . Since there are neither experimental data nor theoretical calculations on the shape of the polarization distribution, we assume for the curves in Fig. 2 that it has the same shape as the distribution of coupling constants  $A_k$ , but with a different width, reflected in the number of nuclear spins  $N_p$  that are appreciably polarized. The motivation for this



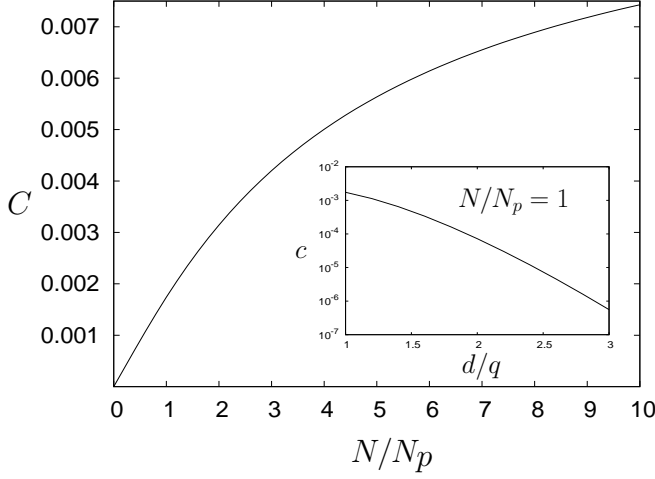


FIG. 2: Numerical prefactor  $c$  (given in Eq. (18)) of the  $t^2$ -term in the decay of the Overhauser field mean value  $\langle h_z(t) \rangle$ . While  $c$  turns out to be independent (for  $N \gtrsim 100$  in the case shown according to numerical summation) of the number  $N$  of nuclear spins within a Bohr radius of the electron envelope wave function, it does depend on the type of structure and the initial polarization. We show the case of a 2-d quantum dot with a Gaussian electron envelope ( $d/q = 1$ ). The dependence on the initial polarization is parameterized by  $N/N_p$ , where  $N_p$  is the number of nuclear spins that is polarized substantially (see text). Inset: dependence of  $c$  on the ratio  $d/q$  for  $N/N_p = 1$ . We see that, e.g., for a donor impurity with a hydrogen-like wave function ( $d/q = 3$ ) the prefactor  $c$  is more than three orders of magnitude smaller compared to the 2-d lateral quantum dot with  $d/q = 1$ .

choice is that if polarization is introduced into the nuclear spin system via electron-nuclear spin flip-flops, the probability for these flip-flops is expected to be proportional to some power of  $A_k/A_0$ . The degree of polarization at the center we denote by  $p \in [-1, 1]$ . We may thus write  $f_k = pe^{-(k/N_p)^{d/d}}$ . We see in Fig. 2 that  $c$  grows monotonically with  $N/N_p$ , i.e., a localized polarization distribution ( $N/N_p > 1$ ) decays more quickly than a wide spread one ( $N/N_p < 1$ ).

In the context of state narrowing,<sup>56,57,58</sup> the short-time behavior of the width of the Overhauser field  $\sigma(t) = \sqrt{\langle h_z^2(t) \rangle - \langle h_z(t) \rangle^2}$  is also of interest. Nuclear spin state narrowing, i.e., the reduction of  $\sigma$ , extends the electron spin decoherence time. Repeating the above calculation for  $\langle h_z^2(t) \rangle$  and using the result for  $\langle h_z(t) \rangle$  we find (up to corrections of  $O(t^4)$ ) for the variance of the Overhauser field

$$\sigma^2(t) = \sigma^2(0) \left( 1 + c_\sigma \frac{t^2}{\tau_e^2} \right), \quad (19)$$

with the range of validity  $\omega \gtrsim A/\sqrt{N}$ , limited by higher-order corrections to the effective Hamiltonian as in the case of  $\langle h_z(t) \rangle$ . Here, the dimensionless constant  $c_\sigma$  is

given by

$$c_\sigma = \frac{1}{16Nc_{\sigma 0}} \sum_{kl} \alpha_k^2 \alpha_l^2 (\alpha_k - \alpha_l) (f_k - f_l) (f_k \alpha_k + f_l \alpha_l), \quad (20)$$

where  $c_{\sigma 0} = \sum_k \alpha_k^2 (1 - f_k^2)$ . Taking the square-root of  $\sigma^2(t)$  and expanding it for  $c_\sigma t^2 / \tau_e^2 \ll 1$  we find for the width (up to corrections of  $O(t^4)$ )

$$\sigma(t) = \sigma(0) \left( 1 + c_\sigma \frac{t^2}{2\tau_e^2} \right). \quad (21)$$

Thus, also for the width of the Overhauser field the initial dynamics is quadratic in time with the same timescale as the mean.

### A. Fully polarized case

In this section we analyze the special case of a fully polarized nuclear spin system, where the effective Hamiltonian derived in Sec. III gives no dynamics and thus the corrections due to the transfer of polarization from the electron to the nuclei become relevant. We thus must return to the full Hamiltonian in Eq. (5). Using the fact that the total spin  $J_z = S_z + \sum_k I_k^z$  is a conserved quantity, we transform into a rotating frame where the Hamiltonian takes the form<sup>27</sup>

$$H' = (\tilde{\epsilon}_z + h_z) S_z + \frac{1}{2} (h_+ S_- + h_- S_+), \quad (22)$$

with  $\tilde{\epsilon} = \epsilon - \eta_z$ . To have any dynamics for a fully polarized nuclear spin system (all spins  $|\uparrow\rangle$ ), the initial state of the electron must be  $s_\downarrow |\downarrow\rangle + s_\uparrow |\uparrow\rangle$ , with  $s_\downarrow \neq 0$ . Since the  $|\uparrow\rangle$  part gives no dynamics we consider  $|\psi(0)\rangle = |\downarrow; \uparrow\uparrow \dots \uparrow\rangle$ . At any later time we may thus write

$$|\psi(t)\rangle = a(t) |\psi(0)\rangle + \sum_k b_k(t) |\uparrow; \uparrow\uparrow \dots \uparrow \downarrow_k \uparrow \dots \uparrow\rangle, \quad (23)$$

with  $a(0) = 1$  and  $b_k(0) = 0, \forall k$ . The same case was studied in Ref. 64. However, this study was performed from the point of view of electron spin decoherence. For the expectation value of  $\langle h_z(t) \rangle$ , we find, in terms of  $a(t)$  and  $b_k(t)$ ,

$$\langle h_z(t) \rangle = \langle \psi(t) | h_z | \psi(t) \rangle = \frac{A}{2} - \sum_k |b_k(t)|^2 A_k, \quad (24)$$

where we have used the normalization condition  $|a(t)|^2 + \sum_k |b_k(t)|^2 = 1$ . Using the time-dependent Schroedinger equation  $i\partial_t |\psi(t)\rangle = H' |\psi(t)\rangle$ , we obtain the differential equations for  $a(t)$  and  $b_k(t)$ :

$$\dot{a}(t) = \frac{i}{4} (2\epsilon_z + A) a(t) - \frac{i}{2} \sum_k b_k(t) A_k, \quad (25)$$

$$\dot{b}_k(t) = -\frac{iA_k}{2} a(t) - \frac{i}{4} (2\epsilon_z + A - 2A_k) b_k(t). \quad (26)$$

Inserting a power-series Ansatz  $a(t) = \sum_l a^{(l)} t^l$  and  $b_k(t) = \sum_l b_k^{(l)} t^l$  into these equations and comparing coefficients yields recursion relations of the form

$$a^{(l+1)} = \frac{i}{4(l+1)} (2\epsilon_z + a) a^{(l)} - \frac{i}{2(l+1)} \sum_k b_k^{(l)} A_k, \quad (27)$$

$$b_k^{(l+1)} = -\frac{i A_k}{2(l+1)} a^{(l)} - \frac{i}{4(l+1)} (2\epsilon_z + A - 2A_k) b_k^{(l)}. \quad (28)$$

Iterating these recursion relations using that  $a(0) = 1$  and  $b_k(0) = 0, \forall k$ , we find, neglecting corrections of  $O(t^4)$ ,

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - \frac{1}{2A} \sum_k A_k^3 t^2. \quad (29)$$

For the case of a 2-d quantum dot with Gaussian envelope wave function, where we have  $A_k = A e^{-k^2/N}/N$ , we find, evaluating  $\sum_k A_k^3$  by turning it into an integral in the continuum limit  $N \gg 1$ , (again up to corrections of  $O(t^4)$ )

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - \frac{1}{6} \left( \frac{t}{\tau_c} \right)^2, \quad (30)$$

where  $\tau_c = N/A$ . To obtain the range of validity for this result we go to higher order in  $t$ . Again for the case of a 2-d quantum dot with Gaussian envelope wave function we find up to  $O(t^4)$ , neglecting terms that are suppressed by  $O(1/N)$  in the  $t^4$ -term,

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - \frac{1}{6} \left( \frac{t}{\tau_c} \right)^2 + \frac{1}{18} \left( \frac{t}{\tau_4} \right)^4. \quad (31)$$

Here,  $\tau_4 = 2\sqrt{N}/\sqrt{A(2\epsilon_z + A)}$ . This shows that in some cases the higher order terms in the short-time expansion can have a considerably shorter timescale. Comparing the short-time expansion with a calculation for  $\langle S_z \rangle$  in the case of uniform coupling constants<sup>64</sup> suggests that the full dynamics contain oscillations with a frequency  $\propto \epsilon_z + A/2$ , thus limiting the range of validity of the short-time expansion to  $t \ll (\epsilon_z + A/2)^{-1}$ .

With this we finish our discussion of the short-time dynamics and of the Zeno effect and move on to long-time behavior. We first show the results of a Dyson-series expansion in Sec. V and in Sec. VI we treat the problem using the generalized master equation, showing that the Dyson-series expansion gives the leading-order contribution in  $A/\omega$ .

## V. DYSON-SERIES EXPANSION

In this section we calculate the expectation value of the Overhauser field  $\langle h_z(t) \rangle$  in a Dyson-series expansion

up to second order in the interaction  $V$ . This allows us to obtain the full time dynamics of  $\langle h_z(t) \rangle$ . Since the Dyson-series expansion is not a controlled expansion (it leads to non-secular divergences in time at higher order), we will only see from the generalized master equation calculation in Sec. VI that the Dyson series result gives the correct leading order contribution in  $A/\omega$ . Thus, the results in this section are expected to be valid in the regime  $\omega \gg A$ .

We transform all operators into the interaction picture by  $\tilde{\mathcal{O}} = e^{iH_0 t} \mathcal{O} e^{-iH_0 t}$ . In the interaction picture we have  $\langle h_z(t) \rangle = \text{Tr}\{\tilde{h}_z \tilde{\rho}(t)\}$ , with  $\tilde{h}_z = h_z$  since  $[H_0, h_z] = 0$ . Expanding  $\tilde{\rho}(t)$  in a Dyson series we find<sup>65</sup>

$$\begin{aligned} \tilde{\rho}(t) = & \rho(0) - i \int_0^t dt' [\tilde{V}(t'), \rho(0)] \\ & - \int_0^t dt' \int_0^{t'} dt'' [\tilde{V}(t'), [\tilde{V}(t''), \rho(0)]] + O(\tilde{V}^3), \end{aligned} \quad (32)$$

where

$$\tilde{V}(t) \equiv e^{iH_0 t} V e^{-iH_0 t} = \frac{S_z}{2\omega} \sum_{k \neq l} e^{iS_z(A_k - A_l)t} I_k^+ I_l^-. \quad (33)$$

We assume again the same initial state as in Sec. IV and thus the term linear in  $\tilde{V}$  will drop out under the trace as it only contains off-diagonal terms. From the remaining two terms we find

$$\begin{aligned} \langle h_z(t) \rangle = & \langle h_z(0) \rangle + \frac{1}{8\omega^2} \sum_{k \neq l} \frac{A_k^2 A_l^2 (f_k - f_l)}{A_k - A_l} \\ & \times \left( \cos \left[ (A_k - A_l) \frac{t}{2} \right] - 1 \right). \end{aligned} \quad (34)$$

We first verify that this result is consistent with the short-time expansion in Sec. IV. For this we use that  $A_k \leq A_0 \propto A/N$  and thus for times  $t \ll \tau_c = N/A$  we may expand the cosine in the above expression, recovering, to second order in  $t$ , the result in Eq. (16). For the full time dynamics we note that the sum over cosines leads to a decay on a timescale of  $\tau_c = N/A$ , since for  $t > \tau_c$  the different cosines interfere destructively. We illustrate this with an example: for a particular choice of the initial polarization distribution ( $d/q = 1$  and  $N_p = N$ ) we may evaluate the sum in Eq. (34) in the continuum limit and find

$$\frac{\langle h_z(t) \rangle}{\langle h_z(0) \rangle} = 1 - \frac{p}{8N} \frac{A^2}{\omega^2} g(t/\tau_c). \quad (35)$$

The function  $g(t)$  is explicitly given by

$$g(t) = \frac{1}{t^4} \left[ t^4 - 16t^2 + 64t \sin\left(\frac{t}{2}\right) - 256 \sin^2\left(\frac{t}{4}\right) \right], \quad (36)$$

with  $g(0) = 0$  and  $g(t \rightarrow \infty) = 1$ . We thus find a power-law decay on a timescale  $\tau_c$  by an amount of  $O(1/N)$ .

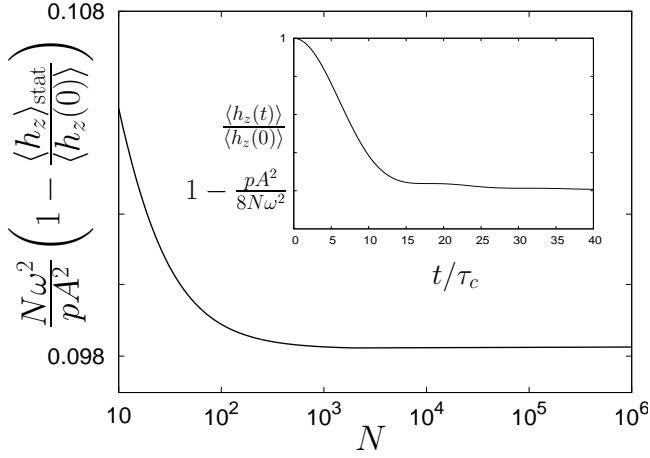


FIG. 3: In this figure we show the  $N$ -dependence of  $1 - \langle h_z \rangle_{\text{stat}} / \langle h_z(0) \rangle$ , i.e., the part by which  $\langle h_z \rangle$  decays in units of  $pA^2/N\omega^2$ , in the regime  $\omega \gg A$ . This plot is for a 3-d defect center with a hydrogen-like electron envelope ( $d/q = 3$ ) and the initial polarization is parameterized by  $N/N_p = 0.5$  as described in Sec. IV. For this choice of polarization distribution the decay is of  $O(1/N)$ . The inset shows the full time dynamics of  $\langle h_z(t) \rangle / \langle h_z(0) \rangle$  as given in Eq.(35) for  $d/q = 1, N \gg 1, N/N_p = 1$ . We see that the decay occurs on a timescale of  $\tau_c = N/A$ .

Since the sum of cosines in Eq. (34) decays, the remaining time-independent sum gives the stationary value (up to the Poincaré recurrence time<sup>66</sup>)

$$\frac{\langle h_z \rangle_{\text{stat}}}{\langle h_z(0) \rangle} = 1 - \left( \frac{A}{\omega} \right)^2 \frac{1}{4N^2 c_0} \sum_{k \neq l} \frac{\alpha_k^2 \alpha_l^2 (f_k - f_l)}{\alpha_k - \alpha_l}. \quad (37)$$

For a system with a large number of nuclear spins  $N \gg 1$  and a sufficiently smooth polarization distribution, this stationary value differs only by a term of  $O(1/N)$  from the initial value, i.e.,  $\langle h_z \rangle_{\text{stat}} / \langle h_z(0) \rangle = 1 - O(1/N)$ . This can be seen in Fig. 3, where we show the  $N$  dependence of  $1 - \langle h_z \rangle_{\text{stat}} / \langle h_z(0) \rangle$ , i.e., the part by which  $\langle h_z \rangle$  decays. The parameters in Fig. 3 are taken for a 3-d defect center with a hydrogen-like electron envelope ( $d/q = 3$ ) and the initial polarization  $N/N_p = 0.5$  as described in Sec. IV. For this choice of polarization distribution the decay is of  $O(1/N)$ . We also find a  $O(1/N)$  behavior for other values of the parameters  $d/q$  and  $N/N_p$  and thus expect this to be generally true for a smoothly varying initial polarization distribution. The inset of Fig. 3 shows the full time dynamics of  $\langle h_z(t) \rangle$  as given in Eq.(35) for  $d/q = 1, N \gg 1, N/N_p = 1$ .

We note that the 4th order of a Dyson series expansion gives secular terms (diverging in  $t$ ). We thus move on to treat the long-time behavior using a master equation approach which avoids these secular terms and shows that the Dyson series result gives the correct leading-order term in  $A/\omega$ .

## VI. GENERALIZED MASTER EQUATION

In this section we study the decay of the Overhauser field mean value  $\langle h_z(t) \rangle$  using the Nakajima-Zwanzig generalized master equation (GME) in a Born approximation. The results in this section are valid in the regime  $\omega \gg A$ , since higher-order corrections to the Born approximation are suppressed by a factor  $(A/\omega)^2$ .

We start from the GME,<sup>66</sup> which for  $P_k \rho(0) = \rho(0)$  reads

$$P_k \dot{\rho}(t) = -iP_k L P_k \rho(t) - \int_0^t dt' P_k L e^{-iQL(t-t')} Q L P_k \rho(t'), \quad (38)$$

where  $L = L_0 + L_V$  is the Liouville superoperator defined as  $(L_0 + L_V)\mathcal{O} = [H_0 + V, \mathcal{O}]$ . The projection superoperator  $P_k$  must preserve  $\langle I_k^z(t) \rangle$  and we choose it to have the form  $P_k = \rho_e(0) \text{Tr}_e \otimes P_{dk} \otimes_{l \neq k} \rho_{I_l}(0) \text{Tr}_{I_l}$  where  $P_{dk}$  projects onto the diagonal in the subspace of nuclear spin  $k$  and is defined as  $P_{dk}\mathcal{O} = \sum_{s=\uparrow, \downarrow} |s_k\rangle \langle s_k| \mathcal{O} |s_k\rangle$ . Further,  $Q = 1 - P_k$ . In a standard Born approximation and using the same initial conditions as above, i.e., a product state and no transverse coherence in the nuclear spin system, we obtain the following integro-differential equation for  $\langle I_k^z(t) \rangle$

$$\begin{aligned} \dot{\langle I_k^z(t) \rangle} = & -\frac{A_k^2}{8\omega^2} \int_0^t d\tau \sum_{l, l \neq k} A_l^2 \cos \left[ \frac{\tau}{2} (A_k - A_l) \right] \\ & \times (\langle I_k^z(t-\tau) \rangle - \langle I_l^z(0) \rangle). \end{aligned} \quad (39)$$

The Born approximation goes to order  $L_V^2$  in the expansion of the self-energy. Higher-order corrections in  $L_V$  are estimated to give contributions to the right-hand side of Eq. (39) that are suppressed by a factor  $(A/\omega)^2$ . We expect the results of this section to be valid at least for  $\omega \gg A$ , although it could in principle happen that (as in the case of the decay of  $\langle S_z(t) \rangle$ <sup>27</sup>) the result for the stationary value has a larger regime of validity. On the other hand it can not be generally excluded that higher-order contributions could dominate at sufficiently long times. Integrating Eq. (39) we find the formal solution

$$\begin{aligned} \langle I_k^z(t) \rangle = & \langle I_k^z(0) \rangle - \frac{A_k^2}{\omega^2} \frac{\alpha_k^2}{8} \int_0^t dt' \int_0^{t'} d\tau \sum_{l, l \neq k} A_l^2 \\ & \times \cos \left[ \frac{\tau}{2} (A_k - A_l) \right] (\langle I_k^z(t'-\tau) \rangle - \langle I_l^z(0) \rangle). \end{aligned} \quad (40)$$

This shows that  $\langle I_k^z(t) \rangle = \langle I_k^z(0) \rangle + O((A/\omega)^2)$  and we may thus iterate this equation and replace  $\langle I_k^z(t'-\tau) \rangle$  in the integral by  $\langle I_k^z(0) \rangle$ . This implies, up to corrections of  $O((A/\omega)^4)$ ,

$$\begin{aligned} \langle I_k^z(t) \rangle = & \langle I_k^z(0) \rangle - \frac{A_k^2}{16\omega^2} \sum_{l, l \neq k} A_l^2 (f_k - f_l) \\ & \times \int_0^t dt' \int_0^{t'} d\tau \cos \left[ \frac{\tau}{2} (A_k - A_l) \right]. \end{aligned} \quad (41)$$

Performing the integrals and summing over the  $\langle I_k^z(t) \rangle$  weighted by their coupling constants  $A_k$ , we recover the Dyson series result in Eq. (34). This shows that the Dyson series expansion gives the leading-order contribution in  $A/\omega$ .

For the analytical solution of Eq. (39) in the stationary limit we perform a Laplace transformation, solve the resulting equation in Laplace space, and calculate the residue of the pole at  $s = 0$  which yields (up to the recurrence time)

$$\begin{aligned} \langle I_k^z \rangle_{\text{stat}} &= \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \langle I_k^z(t) \rangle dt = \lim_{s \rightarrow 0} s \langle I_k^z(s) \rangle \\ &= \frac{1}{Z_k} \sum_l P_k(l) \langle I_l^z(t=0) \rangle, \end{aligned} \quad (42)$$

with  $Z_k = \sum_l P_k(l)$ . We see that  $\langle I_k^z \rangle_{\text{stat}}$  is determined by weighting the neighboring  $\langle I_l^z(t=0) \rangle$  with the probability distribution  $P_k(l)/Z_k$ , which is explicitly given by

$$P_k(l) = \begin{cases} A_l^2 / (A_k - A_l)^2 & : l \neq k, \\ 2\omega^2 / A_k^2 & : l = k. \end{cases} \quad (43)$$

We point out that  $\langle I_k^z \rangle_{\text{stat}}$  can be either smaller or larger than  $\langle I_l^z(t=0) \rangle$  and that  $\sum_k \langle I_k^z \rangle_{\text{stat}} = \sum_k \langle I_l^z(t=0) \rangle$  since the total spin is a conserved quantity. Again expanding the result in Eq.(42) to leading order in  $A/\omega$  and summing over the nuclear spins weighted by their coupling constants  $A_k$ , we recover the same result found in the Dyson series calculation in Eq. (37). Intuitively one would expect a decay even at high fields (although a very slow one) to a state with uniform polarization. The fact that our calculation shows no such decay suggests that the Knight-field gradient, i.e., the gradient in the additional effective magnetic field seen by the nuclei, due to the presence of the electron, is strong enough to suppress such a decay if the flip-flop terms are sufficiently suppressed.

As discussed in Sec. V, the decay to the stationary value occurs on a timescale  $\tau_c = N/A$ . Performing a projective measurement at a time  $t > \tau_m$  resets the initial condition and thus again a small decay occurs. Repeating these measurements at intervals longer than  $\tau_m$  thus allows for a decay of  $\langle h_z(t) \rangle$  to zero.

## VII. CONCLUSION

We have studied the dynamics of the Overhauser field generated by the nuclear spins surrounding a bound electron. We focused our analysis on the effect of the electron-mediated interaction between nuclei due to the hyperfine interaction. At short times we find a quadratic initial decay of the Overhauser field mean value  $\langle h_z(t) \rangle$  on a timescale  $\tau_e = N^{3/2}\omega/A^2$ . Performing repeated strong measurements on  $h_z$  leads to a Zeno effect with the decay changing from quadratic to linear, with a timescale

that is prolonged by a factor  $\tau_e/\tau_m$ , where  $\tau_m$  is the time between consecutive measurements. In Secs. V and VI we have addressed the long-time decay of  $\langle h_z(t) \rangle$  using a Dyson series expansion and a generalized master equation approach. Both show that  $\langle h_z(t) \rangle$  only decays by a fraction of  $O(1/N)$  for a sufficiently smooth polarization distribution and large magnetic field. It remains a subject of further study beyond the scope of this work whether, and on what timescale, the combination of electron-mediated interaction and direct dipole-dipole interaction may lead to a full decay of the Overhauser field. Another interesting question concerns the distribution of nuclear polarization within a quantum dot or defect center and its dependence on the method that is used to polarize the system.

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## APPENDIX A: ESTIMATION OF DIPOLE-DIPOLE CONTRIBUTION

In this appendix we estimate the timescale arising from the direct secular (terms conserving  $I_{z,\text{tot}} = \sum_k I_k^z$ ) dipole-dipole interaction in the short-time expansion of the Overhauser field mean value  $\langle h_z(t) \rangle$ . This gives us the range of validity of our calculation in the main text that only took into account the electron-mediated interaction between nuclei. Let us thus consider the situation where the external magnetic field is very high, such that the electron-mediated flip-flop terms are fully suppressed. In this case the Hamiltonian has the form  $H_{dd} = H_{0,dd} + V_{dd}$ , with

$$H_{0,dd} = \epsilon_z S_z + \eta_z \sum_k I_k^z + S_z h_z - 4 \sum_{k \neq l} b_{kl} I_k^z I_l^z, \quad (A1)$$

$$V_{dd} = \sum_{k \neq l} b_{kl} I_k^+ I_l^-. \quad (A2)$$

Here,  $b_{kl} = \gamma_I^2(3\cos^2(\theta_{kl}) - 1)/r_{kl}^3$ , with  $\theta_{kl}$  being the angle between a vector from nucleus  $k$  to nucleus  $l$  and the  $z$ -axis and  $r_{kl}$  being the distance between the two nuclei.<sup>61</sup> Further,  $\gamma_I$  is the nuclear gyromagnetic ratio. For the short-time expansion, only the off-diagonal terms are relevant, since  $[h_z, H_0] = [\rho(0), H_0] = 0$ . These off-diagonal terms in the case of the electron-mediated interaction are  $S_z \sum_{k \neq l} A_k A_l I_k^+ I_l^- / 2\omega$  (see Eq. (10)). Replacing  $A_k A_l / 2\omega$  by  $b_{kl}$  in the result for the short-time expansion in Eq.(16) and also taking into account the factor of  $1/4$  that comes from  $S_z^2$  in the electron-mediated



case we find

$$\langle h_z(t) \rangle_{\text{dip-dip}} = \langle h_z(0) \rangle - \frac{t^2}{4} \sum_{kl} b_{kl}^2 (A_k - A_l) (f_k - f_l). \quad (\text{A3})$$

To estimate, we restrict the sum to nearest neighbors as the  $b_{kl}$  fall off with the third power of the distance between the two nuclei. Assuming  $f_k = (A_k/A_0)^{N/N_p}$  we find up to corrections of  $O(t^4)$

$$\frac{\langle h_z(t) \rangle_{\text{dip-dip}}}{\langle h_z(0) \rangle} \approx 1 - \frac{t^2}{\tau_d^2}, \quad \tau_d = \frac{\sqrt{N_p N}}{b}, \quad (\text{A4})$$

with  $b$  being the nearest-neighbor dipole-dipole coupling. For GaAs we have  $b \sim 10^2 \text{ s}^{-1}$  (with  $\gamma_I \approx 10 \text{ MHz/T}^{67}$ ). For  $NN_p \gg 1$  we have  $\tau_d \gg 10^{-2} \text{ s}$ . In the magnetic field range shown in Table I we thus have  $\tau_d \gg \tau_e/\sqrt{c}$ , which justifies neglecting the direct dipole-dipole coupling in the short-time expansion.

## APPENDIX B: MEASUREMENT ACCURACY

The description of the Zeno effect in Sec. II relied on the assumption that the measurements on  $h_z$  set all off-diagonal elements of the density matrix to zero. This assumption requires on one hand a perfect measurement accuracy for  $h_z$  (we discuss deviations from that below), but on the other hand it also requires the  $h_z$ -eigenstates to be non-degenerate. For non-degenerate  $h_z$  eigenstates a measurement of  $h_z$  fully determines the polarization distribution  $f_k$  and we may thus write  $\rho_I$  after the measurement again as a direct product with  $\rho_{I_k}(\tau_m) = 1/2 + f_k(\tau_m)I_k^z$ . After the measurement, we thus again have the same time evolution for  $\langle h_z(t) \rangle$  as given in Eq. (16), but with  $f_k$  replaced by  $f_k(\tau_m)$ . Iterating Eq. (16) for the case of  $m$  consecutive measurements at intervals  $\tau_m$  one obtains Eq. (3).

Instead of the idealized assumption of a projective measurement we now allow for imperfect measurements. To describe these measurements we use a so-called POVM (positive operator valued measure).<sup>51</sup> In a general POVM measurement the density matrix changes accordingly to<sup>51</sup>

$$\rho \rightarrow \rho' = \int \sqrt{F_y} \rho \sqrt{F_y} dy, \quad (\text{B1})$$

when averaging over all possible measurement outcomes  $y$ . The probability to measure outcome  $y$  is given by  $P(y) = \text{Tr}\{\rho F_y\}$  and the condition  $\int dy F_y = 1$  ensures that the probabilities sum to unity. We consider the nuclear density matrix  $\rho_I$  in a basis of  $h_z$  eigenstates  $|n\rangle$  with  $h_z |n\rangle = h_z^n |n\rangle$ . We denote the matrix elements of  $\rho_I$  by  $\rho_I(n, m) = \langle n | \rho_I | m \rangle$ . For the following description we assume that the diagonal of the nuclear spin density matrix before the measurement is Gaussian distributed around its mean value  $\langle h_z \rangle$  with a width  $\sigma$ , i.e.,

$$\rho_I(n, n) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{(h_z^n - \langle h_z \rangle)^2}{2\sigma^2}\right]. \quad (\text{B2})$$

For an unpolarized equilibrium (infinite temperature) state, the width is  $\sigma \propto A/\sqrt{N}$ . Here,  $\sigma$  can take any value. Let us now consider a measurement that determines the value of  $h_z$  up to an accuracy  $\eta$  with a Gaussian lineshape. We refer to  $\eta$  as the measurement accuracy. If the outcome of the measurement is  $\langle h_z \rangle + y$ , the diagonal of the nuclear spin density matrix after a measurement has the form

$$\rho'_I(n, n; y) = \frac{1}{\sqrt{2\pi}\eta} \exp\left[-\frac{(h_z^n - \langle h_z \rangle - y)^2}{2\eta^2}\right]. \quad (\text{B3})$$

Since we aim to describe measurements that at least partially project the nuclear spin state, we have  $\eta < \sigma$ . The POVM that describes such a measurement is given by

$$F_y = \sum_n f(n, y) |n\rangle \langle n|, \quad (\text{B4})$$

with

$$f(n, y) = \frac{\sigma}{\eta \sqrt{2\pi(\sigma^2 - \eta^2)}} \exp\left[-\frac{(h_z^n - \langle h_z \rangle - y)^2}{2\eta^2}\right] \times \exp\left[-\frac{(h_z^n - \langle h_z \rangle)^2}{2\sigma^2} - \frac{y^2}{2(\sigma^2 - \eta^2)}\right]. \quad (\text{B5})$$

We note that for  $\eta \ll \sigma$  we have  $f(n, y) \approx \exp(-(h_z^n - \langle h_z \rangle - y)^2/2\eta^2)/\sqrt{2\pi}\eta$ . With  $f(n, y)$ , the operators  $F_y$  are fully determined and it is straightforward to calculate the probability for obtaining the measurement result  $\langle h_z \rangle + y$

$$P(y) = \frac{1}{\sqrt{2\pi(\sigma^2 - \eta^2)}} \exp\left[-\frac{y^2}{2(\sigma^2 - \eta^2)}\right]. \quad (\text{B6})$$

Clearly, the probabilities add up to one ( $\int P(y) dy = 1$ ) as they should. Also, when weighting the  $\rho'_I(n, n; y)$  with their probabilities for occurring, we find  $\int \rho'_I(n, n; y) P(y) dy = \rho_I(n, n)$ . Using Eq. (B1) we thus find for the matrix elements after a measurement, when averaging over all possible measurement outcomes

$$\rho_I(n, m) \rightarrow \rho'_I(n, m) = \rho_I(n, m) \int \sqrt{f(n, y) f(m, y)} dy, \quad (\text{B7})$$

with (for  $\eta \ll \sigma$ )

$$f(n, y) \approx \frac{1}{\sqrt{2\pi}\eta} \exp\left[-\frac{(h_z^n - h_{z0} - y)^2}{2\eta^2}\right]. \quad (\text{B8})$$

Again, for  $\eta \ll \sigma$ , we thus have

$$\rho'_I(n, m) = \rho_I(n, m) \exp\left[-\frac{(h_z^n - h_z^m)^2}{8\eta^2}\right]. \quad (\text{B9})$$

To reduce the off-diagonal elements, the measurement accuracy must be better than the difference in eigenvalues. In the limit  $\eta \rightarrow 0$  a projective measurement is recovered, which sets all off-diagonal elements to zero. Up to  $t^2$  in the short-time expansion, only off-diagonal elements between states that differ at most by two flip-flops can

become non-zero. Thus, to have at least a partial Zeno effect,<sup>52</sup> resulting from the off-diagonal elements being partially reduced, the requirement on the measurement accuracy is  $\eta \lesssim h_z^n - h_z^m$  with  $|n\rangle = I_k^+ I_l^- I_p^+ I_q^- |m\rangle$ . For coupling constants  $A_k = Ae^{-k/N}/N$ , we have typically  $h_z^n - h_z^m \propto A/N^{3/2}$ . Besides destroying the off-

diagonal elements of  $\rho_I$  through a measurement, there are also “natural” dephasing mechanisms, such as inhomogeneous quadrupolar splittings, electron-phonon coupling, or spin-lattice relaxation, that can lead to a reduction of the off-diagonal elements of  $\rho_I$ .

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